- **Quantifier scope reduction and expansion equivalences**: To adjust the scope of quantifiers (∀, ∃) while preserving logical equivalence.
  - Let $A(x)$ be a formula in which x appears freely, and let B be a formula in which x does not appear.

Universal quantifier	Existential quantifier
$\forall x(A(x) \lor B) \Leftrightarrow \forall x A(x) \lor B$	$\exists x(A(x) \lor B) \Leftrightarrow \exists x A(x) \lor B$
$\forall x(A(x) \land B) \Leftrightarrow \forall x A(x) \land B$	$\exists x(A(x) \land B) \Leftrightarrow \exists x A(x) \land B$
$\forall x(A(x) \to B) \Leftrightarrow \exists x A(x) \to B$	$\exists x(A(x) \to B) \Leftrightarrow \forall x A(x) \to B$
$\forall x(B \to A(x)) \Leftrightarrow B \to \forall x A(x)$	$\exists x(B \to A(x)) \Leftrightarrow B \to \exists x A(x)$

- **Substitution Rule:**
  Let $\Phi(A)$ be a formula containing formula A, and $\Phi(B)$ be the formula obtained by replacing all occurrences of A in $\Phi(A)$ with formula B. Then, $\Phi(A) \Leftrightarrow \Phi(B)$.
- **Renaming Rule:**
  In a formula A, change the bound variables (and their occurrences within the scope of the quantifier) of a quantifier to an individual term that has not appeared within the scope of that quantifier. The rest of the formula remains unchanged, and the resulting formula is denoted as A'. Then, $A' \Leftrightarrow A$.

Note:

(1) ***Substitution*** can be used to transform expressions and find equivalent forms of expression.

(2) ***Renaming*** can eliminate variable name conflicts and clarify the scope of quantifiers.

(3) When substituting, the replaced terms should not become variables within the scope of some quantifier.

(4) When renaming, only the variable names bound by quantifiers are changed, and the rest of the formula structure remains unchanged.

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e.g. **Example:** Eliminate the individual variables that appear both in the constraints and as free variables in the equation.

(1) $\forall xF(x,y,z) \rightarrow \exists yG(x,y,z)$

$\Leftrightarrow \forall uF(u,y,z) \rightarrow \exists yG(x,y,z)$

$\Leftrightarrow \forall uF(u,y,z) \rightarrow \exists vG(x,v,z)$ （Renaming Rule Equivalence)

Avoid variable confusion and improve expression readability and consistency.

(2) $\forall x(F(x,y) \rightarrow \exists yG(x,y,z))$

$\Leftrightarrow \forall x(F(x,y) \rightarrow \exists tG(x,t,z))$ （Renaming Rule Equivalence）

Only changed the bound variable name of the existential quantifier y.

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e.g. **Example:** Let the domain of individuals be $D=\{a,b,c\}$, eliminate the quantifiers in the following formula:

(1) $\forall x(F(x)\rightarrow G(x))$

$\Leftrightarrow (F(a)\rightarrow G(a))\wedge(F(b)\rightarrow G(b))\wedge(F(c)\rightarrow G(c))$

(2) $\forall x(F(x)\vee\exists yG(y))$

$\Leftrightarrow \forall xF(x)\vee\exists yG(y)$ （Quantifier scope reduction）

$\Leftrightarrow(F(a)\wedge F(b)\wedge F(c))\vee(G(a)\vee G(b)\vee G(c))$

(3) $\exists x\forall yF(x,y)$

$\Leftrightarrow \exists x(F(x,a)\wedge F(x,b)\wedge F(x,c))$

$\Leftrightarrow (F(a,a)\wedge F(a,b)\wedge F(a,c))\vee(F(b,a)\wedge F(b,b)\wedge F(b,c))$

$\vee(F(c,a)\wedge F(c,b)\wedge F(c,c))$

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e.g. Example: Given *I*: (a) $D=\{2,3\}$, (b) $\bar{f}: \bar{f}(2)=3, \bar{f}(3)=2$,

(c) $\bar{F}(x)$: x is even, $\bar{G}(x,y)$: $x=2 \vee y=2$, $\bar{L}(x,y)$: $x=y$.

Solve the true value under *I*:

(1) $\exists x(F(f(x)) \wedge G(x, f(x)))$

Solve: $(F(f(2)) \wedge G(2, f(2))) \vee (F(f(3)) \wedge G(3, f(3)))$

$\Leftrightarrow (0 \wedge 1) \vee (1 \wedge 1) \Leftrightarrow 1$

(2) $\exists x \forall y L(x,y)$

Solve: $\forall y L(2,y) \vee \forall y L(3,y)$

$\Leftrightarrow (L(2,2) \wedge L(2,3)) \vee (L(3,2) \wedge L(3,3))$

$\Leftrightarrow (1 \wedge 0) \vee (0 \wedge 1) \Leftrightarrow 0$

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e.g. **Example:** Prove the following equivalence:

$$\neg\exists x(M(x)\wedge F(x)) \Leftrightarrow \forall x(M(x)\rightarrow \neg F(x))$$

Prove:

Left $\Leftrightarrow \forall x\, \neg(M(x)\wedge F(x))$ （De Morgan's laws for quantifiers）

$\Leftrightarrow \forall x(\neg M(x)\vee\neg F(x))$

$\Leftrightarrow \forall x(M(x)\rightarrow \neg F(x))$

i The proof applies three key logical transformation rules: **quantifier negation**, **De Morgan's laws**, and **implication equivalence**.

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- 3.2.1 First-Order Logic Equivalences and Substitution Rules

  Basic Equivalences Substitution Rules, Renaming Rules

- 3.2.2 Prenex normal form of first-order logic

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- **Definition 3.8:** Let A be a first-order logic formula. If A has the form

$$Q_1x_1Q_2x_2\ldots Q_kx_k\, B$$

where each Q_i is either ∀ or ∃ (for $1 \le i \le k$), and B is a formula without quantifiers, then A is called a ***prenex normal form***.

e.g. **Examples:**

$\forall x \exists y (F(x) \to (G(y) \land H(x,y)))$	(prenex normal form)
$\forall x \neg (F(x) \land G(x))$	(prenex normal form)
$\forall x (F(x) \to \exists y (G(y) \land H(x,y)))$	(not prenex normal form)
$\neg \exists x (F(x) \land G(x))$	(not prenex normal form)

Objective：

Key Concepts：

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